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A MIXED AUTOREGRESSIVE-MOVING AVERAGE EXPONENTIAL SEQUENCE AND POINT PROCESS (EARMA 1,1)

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Naval Postgraduate School
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and
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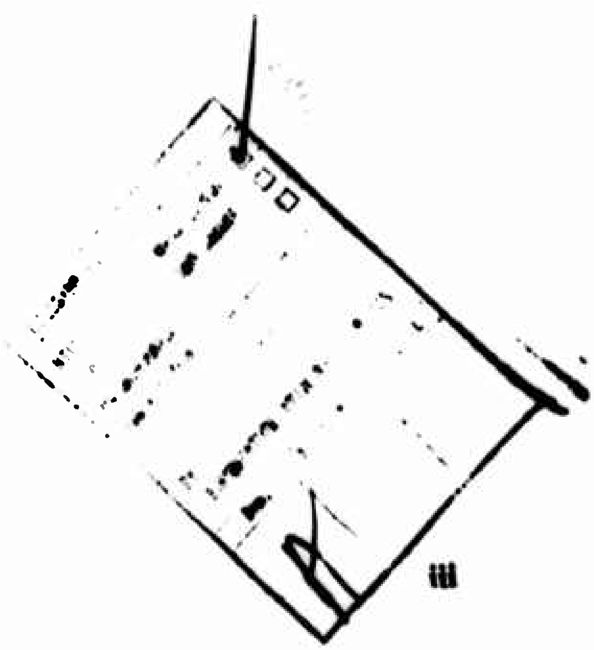

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A stationary sequence of random variables with exponential marginal distributions and the correlation structure of an ARMA (1,1) process is defined. The process is formed as a random linear combination of i.i.d. exponential random variables and is very simple to generate on a computer. Moments and joint distributions for the sequence are obtained, as well as limiting properties of sums of the random variables and of the point (over)

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process whose intervals have the EARMA (1,1) structure.

1. INTRODUCTION

We discuss the stationary sequence of random variables $\{X_n\}$ which is formed from a sequence $\{\epsilon_n\}$ of independent random variables, each with the same exponential distribution with parameter $\lambda > 0$, according to the probabilistic linear model

$$X_n = \begin{cases} \beta\epsilon_n & \text{with probability } \beta, \\ \beta\epsilon_n + A_{n-1} & \text{with probability } (1-\beta), \end{cases} \tag{1.1}$$

for $n = 1,2,\ldots,$ where $0 \le \beta \le 1$, and

$$A_n = \begin{cases} \rho A_{n-1} & \text{with probability } \rho, \\ \rho A_{n-1} + \epsilon_n & \text{with probability } (1-\rho), \end{cases} \tag{1.2}$$

for $n = 1,2,\ldots$ and $0 \le \rho \le 1$. The random variable A_0 has an arbitrary distribution but, if it is taken to be exponential with parameter λ, then the X_n's, $n = 1,2,\ldots,$ form a stationary sequence of dependent exponential random variables.

The model of (1.1) and (1.2) will be called EARMA(1,1), (exponential autoregressive-moving average, each of order 1); it extends the moving average model of Lawrence and Lewis (1975), which is essentially (1.1) with A_{n-1} replaced by ϵ_{n-1}, and the autoregressive model of Gaver and Lewis (1975), which is just (1.2), by combining the two basic structures defined in those papers. Further extensions are given by Lewis (1975).

The model defined by (1.1) and (1.2) is actually a backward model; an equivalent forward model can be defined similarly.

However, the two, while similar, are not equivalent. This is because the sequences are not time reversible in the sense that $\{X_1,\ldots,X_k\}$ does not have the same joint probability distribution as $\{X_{-1},\ldots,X_{-k}\}$. The properties of one model can be derived by the same techniques as those of the other, so we consider only the backward model. The time dependence does, however, come in in an essential way in estimation problems for the model.

Let $T_r = X_1 + \ldots + X_r$. Then T_r can be thought of as the time of occurrence of the $r^{\underline{th}}$ event in a point process having $\{X_i\}$ as the interval sequence. Further let N_t be the number of events that occur in $(0,t]$ in the point process.

Various moments and joint distributions for the $\{X_i\}$ sequence are obtained in the next two sections. A recursive scheme for obtaining the Laplace transform of T_r is obtained in Section 4. The variance time curve and sequence are then obtained. In Section 6 the asymptotic behavior of the sequence $\{X_n\}$ is studied and limit theorems for T_n, N_t and $E[N_t]$ are given.

Finally, several extensions of the model are discussed as well as another model having similar correlation structure.

2. SOME PRELIMINARY PROPERTIES OF THE EARMA (1,1) MODEL

Let $\{U_n\}$ and $\{V_n\}$ be independent sequences of independent random variables taking the values $\{0,1\}$ with $P\{U_n = 0\} = \beta$ and $P\{V_n = 0\} = \rho$. Then we may write (1.1) and (1.2) as

$$X_n = \beta\epsilon_n + U_n A_{n-1}, \tag{2.1}$$

with

$$A_n = \rho A_{n-1} + V_n\epsilon_n \tag{2.2}$$

for $n = 1,2,\ldots,$ where $\{\epsilon_n\}$ is a sequence of independent exponential random variables with parameter λ. Unless otherwise indicated we will assume that $A_0 = \epsilon_0$; that is, A_0 has exponential distribution with parameter λ and is independent of $\{\epsilon_n;\ n = 1,2,\ldots\}$, $\{U_n\}$, and $\{V_n\}$.

It is not hard to show that X_n has an exponential distribution with parameter λ; in fact by (2.2), for $s \geq 0$

$$E\{\exp(-sA_n)\} = E\{\exp(-s\rho A_{n-1})\}\left(\frac{\lambda+s\rho}{\lambda+s}\right)$$

for $n = 1,2,\ldots$. By induction (Gaver and Lewis, 1975),

$$E[\exp\{-sA_n\}] = \frac{\lambda}{\lambda+s} \quad \text{for} \quad n = 0,1,2,\ldots$$

since $A_0 = \epsilon_0$. Hence

$$\begin{aligned} E\{\exp(-sX_n)\} &= E[\exp\{-s(\beta\epsilon_n + U_n A_{n-1})\}] \\ &= \beta\,\frac{\lambda}{\lambda+s\beta} + (1-\beta)\frac{\lambda}{\lambda+s\beta}\,\frac{\lambda}{\lambda+s} \\ &= \frac{\lambda}{\lambda+s}, \end{aligned}$$

showing that the marginal distribution of the X_n's, like those of the ϵ_n's, is exponential with parameter λ.

However, the X_n's are not independent, as seen by the following calculation of the covariance between X_n and X_{n+k}. Ignoring terms which cancel, we get

$$C_{1,1}(k) = E(X_n X_{n+k}) - E(X_n)E(X_{n+k})$$

$$= \beta(1-\beta)\{E(\epsilon_n A_{n+k-1}) - E(\epsilon_n)E(A_{n+k-1})\}$$

$$+ (1-\beta)^2\{E(A_{n-1}A_{n+k-1}) - E(A_{n-1})E(A_{n+k-1})\}.$$

By induction arguments we get

$$E(\epsilon_n A_{n+k}) - E(\epsilon_n)E(A_{n+k}) = \rho^k(1-\rho)\frac{1}{\lambda}$$

and

$$E(A_n A_{n+k}) - E(A_n)E(A_{n+k}) = \rho^k \frac{1}{\lambda}.$$

Therefore, the serial correlation $\rho(k) = C_{1,1}(k)/\mathrm{Var}(x)$ is

$$\rho(k) = \mathrm{corr}(X_n, X_{n+k}) = \rho^{k-1}c(\beta,\rho), \tag{2.3}$$

where

$$c(\beta,\rho) = \beta(1-\beta)(1-\rho) + (1-\beta)^2\rho \tag{2.4}$$

$$= \beta - 3\beta\rho + 2\beta^2\rho - \beta^2 + \rho$$

$$= \beta(1-\beta) + \rho(1-\beta)(1-2\beta).$$

When $\rho = 0$ we have the correlation for the EMA1 model given by Lawrence and Lewis (1975). When $\beta = 0$ we get the correlation for the autoregressive model EAR1 given by Gaver and Lewis (1975). By (2.3) and (2.4) the first order correlation is nonnegative and bounded above by 1. Note that, if

$\frac{\partial}{\partial\beta} c(\beta,\rho) = 1 - 3\rho - 2\beta(1-2\rho) = 0$, then $\beta = \frac{1-3\rho}{2(1-2\rho)}$. If $\frac{1}{3} \le \rho < \frac{1}{2}$ this value of β is non-positive so that, for fixed ρ, $c(\beta,\rho)$ decreases monotonically from the value ρ at $\beta = 0$ to zero at $\beta = 1$. If $\frac{1}{2} < \rho \le 1$, then $\frac{1-3\rho}{2(1-2\rho)} \ge 1$. Finally, $\frac{\partial}{\partial\beta} c(\beta,\frac{1}{2}) < 0$. Hence, for fixed ρ, as a function of β $c(\beta,\rho)$, which always is equal to ρ at $\beta = 0$ and equal to 0 at $\beta = 1$, is single valued on $[0,1]$ for $\rho \ge \frac{1}{3}$. For $\rho < \frac{1}{3}$ it is double valued on $[0,1]$. This result will be useful in the estimation of ρ and β. In fact if $\rho \ge \frac{1}{3}$ estimates of ρ and β can be obtained from the first two serial correlations. For $\rho < \frac{1}{3}$ higher order joint moments are needed.

Note that the second order joint moments or correlation structure is that of the so-called ARMA (1,1) model (cf. Box and Jenkins, 1971); consequently the spectrum will be

$$f_+(\omega) = \frac{1}{\pi}\{1 + 2 \sum_{k=1}^{\infty} \rho(k)\cos(k\omega)\}$$

$$= \frac{1}{\pi}\left[\frac{1 + \{1-2c(\beta,\rho)\}\rho^2 - 2\rho\{1-c(\beta,\rho)\cos\omega\}}{1 + \beta^2 - 2\rho\cos\omega}\right] \qquad (2.5)$$

for $0 \le \omega \le \pi$. The spectrum has the constant value $\frac{1}{\pi}$ when the X_n's are independent, $(\rho = \beta = 0;\ \beta = 1)$.

Note that the correlations $\rho(k)$ are all positive in this EARMA (1,1) model, unlike the ARMA (1,1) model. This seems to be the greatest limitation of the model.

3. HIGHER-ORDER JOINT MOMENTS

We now proceed to the calculation of other joint moments for $\{X_n\}$. These are useful in estimating β, since it is not possible, when $\rho < \frac{1}{3}$, to distinguish between the case $\{\beta,\rho\}$ and $\{(1-\beta),\rho\}$ on the basis of the second order joint moments. This is closely related to the question of time invariance discussed in the Introduction; the time (or serial number) dependence shows up clearly in higher order joint moments.

First, for $k \geq 1$, eliminating terms that cancel, we have

$$C_{2,1}(k) = E(X_n^2 X_{n+k}) - E(X_n^2)E(X_{n+k})$$

$$= \beta^2(1-\beta)\{E(\epsilon_n^2 A_{n+k-1}) - E(\epsilon_n^2)E(A_{n+k-1})\}$$

$$+ 2\beta(1-\beta)\{E(\epsilon_n A_{n-1} A_{n+k-1}) - E(\epsilon_n A_{n-1})E(A_{n+k-1})\}$$

$$+ (1-\beta)^2\{E(A_{n-1}^2 A_{n+k-1}) - E(A_{n-1}^2)E(A_{n+k-1})\}$$

$$= \rho^{k-1}[E(\epsilon_n)\mathrm{Var}(\epsilon_n)2\beta(1-\beta)^2$$

$$+ \{E(\epsilon_n^3) - E(\epsilon_n^2)E(\epsilon_n)\}\{\beta^2(1-\beta)(1-\rho) + (1-\beta)^2\rho\}]$$

$$= \rho^{k-1}\frac{1}{\lambda^3}[2\beta(1-\beta)^2 + 4\{\beta^2(1-\beta)(1-\rho) + (1-\beta)^2\rho\}]. \quad (3.1)$$

When $\rho = 0$, so that we have just the first order moving average process, (3.1) becomes for $k = 1$

$$C_{2,1}(1) = \frac{1}{\lambda^3}\{2\beta(1-\beta)(1+\beta)\} = \frac{1}{\lambda^3}\{2\ \rho(1)\ (1+\beta)\}.$$

It is useful to write the multiplier of ρ^{k-1}/λ^3 in (3.1) as a polynomial in ρ and a polynomial in β and we then have

$$C_{2,1}(k) = \rho^{k-1}\frac{1}{\lambda^3}(1-\beta)\{2\beta(1+\beta)+\rho(1-\beta-\beta^2)\}$$

$$= \rho^{k-1}\frac{1}{\lambda^3}\{\rho+\beta(2-2\rho)+\beta^3(\rho-2)\}.$$

Similarly we get for

$$C_{1,2}(k) = E(X_n X^2_{n+k}) - E(X_n)E(X^2_{n+k})$$

$$= \rho^{k-1}\frac{2}{\lambda^3}[\{\beta(1-\beta)(1-\rho)+(1-\beta)^2\rho\}(1+\beta) \tag{3.2}$$

$$+ \rho^{k-1}\{\rho^2+\beta(1-3\rho^2)+\beta^2(2\rho^2-1)$$

$$= \frac{2}{\lambda}[C_{1,1}(k)(1+\beta)+\frac{\rho^{2(k-1)}}{\lambda^2}\{\rho^2+\beta(1-3\rho^2)+\beta^2(2\rho^2-1). \tag{3.3}$$

Again (3.3) can also be written as polynomials in β and ρ. When $k = 1$ we get

$$\frac{\lambda^3}{2}C_{1,2}(1) = (1-\beta)\{\beta(2+\beta)+\rho(1+\beta)(1-2\beta)+\rho^2(1-2\beta)\} \tag{3.4}$$

$$= \rho(1+\rho)+\beta\{2(1-\rho)-3\rho^2\}+\beta^2(-1-\rho+2\rho^2)+\beta^3(2\rho-1). \tag{3.5}$$

The fact that $C_{2,1}(k) \neq C_{1,2}(k)$ indicates that the $\{X_n\}$ sequence is not time reversible. This can hopefully be exploited, as in the EMA1 process, to estimate β, and, in particular, differentiate between the cases ρ,β and $\rho,(1-\beta)$. Higher order moments, e.g. $C_{2,2}(k)$, are useful in deriving the asymptotic variances of sample serial correlations for the model. This will be discussed elsewhere.

The above third-order joint moments are special cases of the third order joint moments with two lags, j and k, whose double Fourier transform will give the bispectrum of the $\{X_n\}$ sequence.

For $j \geq 1$, $k \geq 1$, and $\rho < 1$, similar calculations to those above show that

$$
\begin{aligned}
C_{1,1,1}(j,k) &= E(X_n X_{n+j} X_{n+j+k}) - E(X_n)E(X_{n+j})E(X_{n+j+k}) \\
&= \frac{5}{\lambda^3}\{\beta(1-\beta)^2\rho^k\rho^{2(j-1)}(1-\rho) + (1-\beta)^3\rho^k\rho^{2j}\} \qquad (3.6) \\
&+ \frac{1}{\lambda^3}\{\beta(1-\beta)^2[\rho^k\{\rho^{2j} + \frac{[1-\rho^{2(j-1)}]}{1+\rho} \\
&+ 2\rho^{2j-1}(1-\rho) + 2\rho^{j-1}(1-\rho)(1-\rho^{j-1})+\rho^j\} \\
&+ \rho^{k-1}\{(1-\rho)^2\rho^{j-1} + (1-\rho)\} \\
&+ \rho^{j-1}\{(1-\rho)[1-\rho^{k-1}] + \rho\}] \\
&+ \beta^2(1-\beta)[\rho^k[\rho^{j-1}(1-\rho)] + \rho^{k-1}(1-\rho) + \rho^{j-1}(1-\rho)] \\
&+ (1-\beta)^3[\rho^k\{\frac{1-\rho^{2j}}{1+\rho} + 2\rho^j(1-\rho^j)\} \\
&+ \rho^{k-1}\{(1-\rho)\rho^j + \rho^j(1-\rho^{k-1})\}]\}.
\end{aligned}
$$

We give this expression for completeness and because it is clear that second-order joint moments do not describe completely a process which is as non-normal as the EARMA(1,1) described here. It is felt, however, that the special cases (3.1) and (3.2) of the third order moment $C_{1,1,1}(j,k)$ when $j = 0$ and $k = 0$ respectively will give all necessary information.

It is also possible to derive Laplace Stieltjes transforms for the joint distributions of several X_i's. These will be multivariate exponential distributions. Thus we have, for example,

$$E[\exp\{-s_1X_n - s_2X_{n+1}\}] =$$
$$\left(\frac{\lambda}{\lambda+s_1\beta}\right)\left(\frac{\lambda}{\lambda+s_2\beta}\right)\left[\frac{\beta(\lambda+\beta s_1)}{(\lambda+s_1)} + \frac{\lambda(1-\beta)(\lambda+\beta s_1+\rho s_2)^2}{(\lambda+s_2\rho)(\lambda+s_1+s_2\rho)(\lambda+s_1\beta+s_2)}\right]. \quad (3.7)$$

Despite its relatively simple form, this transform does not lend itself to easy derivation of moments, e.g. $E[X_n^2X_{n+1}]$, $E[X_nX_{n+1}^2]$, or $E[X_n^2X_{n+1}^2]$, or of conditional moments, as for the EMA1 case ($\rho = 0$) in Lawrance and Lewis (1975); nor is it invertible to give approximate likelihoods for the process.

When $s_1 = s_2 = s$ we get the transform of the sum $X_n + X_{n+1}$ as

$$E[\exp\{-s(X_n+X_{n+1})\}] =$$
$$\left(\frac{\lambda}{\lambda+s\beta}\right)^2\left[\frac{\beta(\lambda+\beta s)}{(\lambda+s)} + \frac{\lambda(1-\beta)(\lambda+\beta s+\rho s)^2}{(\lambda+\rho s)\{\lambda+s(1+\rho)\}(\lambda+s+s\beta)}\right]. \quad (3.8)$$

Transforms of the distributions of the sums of adjacent X_i's are very useful in the point process theory of the model; if these can be obtained one can obtain the second order properties of the counting function of the point process in the form of either an intensity function, the (Bartlett) spectrum of counts or the variance time curve (see Cox and Lewis, 1966, Ch. 4). These transforms are discussed in the next section.

4. THE SEQUENCE $\{T_r\}$.

Recall that $T_r = X_1 + \ldots + X_r$. We will interpret T_r as the time of the $r^{\underline{th}}$ event in a point process which starts with an event at the origin. We will obtain a recursive relationship for computing $E\{\exp(-sT_r)\}$ for $r = 1,2,\ldots$ and $s \geq 0$. Let

$$\psi(s_1,s_2) = E[\exp\{-s_1T_1 - s_2A_1\}]. \tag{4.1}$$

Then, by direct computation from the definition (1.1) and (1.2)

$$\psi(s_1,s_2) = \frac{\{\lambda(\lambda+s_1\beta+s_2\rho)\}^2}{(\lambda+s_1\beta+s_2)(\lambda+s_1\beta)(\lambda+s_2\rho)(\lambda+s_2\rho+s_1)} . \tag{4.2}$$

Now we define

$$\begin{aligned} b(s_1,s_2) &= E[\exp\{-(s_1\beta+s_2V_1)\epsilon_1\}] \\ &= \frac{\lambda(\lambda+s_1\beta+s_2\rho)}{(\lambda+s_1\beta+s_2)(\lambda+s_1\beta)} \end{aligned} \tag{4.3}$$

and let

$$\psi_r(s_1,s_2) = E[\exp\{-s_1T_r - s_2A_r\}].$$

Then, for $r \geq 2$

$$\begin{aligned} \psi_r(s_1,s_2) &= E[\exp\{-s_1(T_{r-1}+\beta\epsilon_r+U_rA_{r-1}) - s_2(\rho A_{r-1}+V_r\epsilon_r)\}] \\ &= E[\exp\{-(s_1\beta+s_2V_r)\epsilon_r - s_1T_{r-1} - (s_1U_r+s_2\rho)A_{r-1}\}] \\ &= b(s_1,s_2)E\{\psi_{r-1}(s_1,s_1U_r+s_2\rho)\} \\ &= b(s_1,s_2)[\beta\psi_{r-1}(s_1,s_2\rho) + (1-\beta)\psi_{r-1}(s_1,s_1+s_2\rho)]. \end{aligned} \tag{4.4}$$

For $\beta = 0$ and $\rho = 0$ we get, respectively, the recursion relationships for the EAR1 process (Gaver and Lewis, 1975) and the EMA1 process (Lawrence and Lewis, 1975) which have explicit solutions which lead to expressions for $E\{\exp(-sT_r)\}$.

Using (4.4) we can calculate the Laplace transform of T_r recursively. In particular we have for T_1 that, using (4.1)

$$\psi(s,0) = E[\exp\{-sT_1\}] = \frac{\lambda}{\lambda+s}\,, \tag{4.5}$$

as it should. Then we get

$$\psi_2(s,0) = E[\exp\{-sT_2\}]$$
$$= \frac{\lambda}{\lambda+s\beta}\left[\beta\,\frac{\lambda}{\lambda+s} + (1-\beta)\,\frac{\{\lambda(\lambda+s\beta+s\rho)\}^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\rho)(\lambda+s\rho+s)}\right] \tag{4.6}$$

which agrees with (3.8). Unfortunately the expressions become very unwieldy as r increases. We have

$$\psi_3(s,0) = \left(\frac{\lambda}{\lambda+s\beta}\right)^2\left[\beta^2\frac{\lambda}{\lambda+s} + \beta(1-\beta)\left\{\frac{\{\lambda(\lambda+s\beta+s\rho)\}^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\rho)(\lambda+s+s\rho)}\right.\right.$$
$$\left. + \frac{\{\lambda(\lambda+s\beta+s\rho^2)\}^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\rho^2)(\lambda+s+s\rho^2)}\right\}$$
$$\left. + (1-\beta)^2\,\frac{(\lambda+s\beta+s\rho)\{\lambda(\lambda+s\beta+s\rho+s\rho^2)\}^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho)(\lambda+s\rho+s\rho^2)(\lambda+s+s\rho+s\rho^2)}\right]. \tag{4.7}$$

For the transform of the distributions of T_4 and T_5 we get

$$\psi_4(s,0) = \left(\frac{\lambda}{\lambda+s\beta}\right)^3\left[\beta^3\,\frac{\lambda}{\lambda+s} + \beta^2(1-\beta)\sum_{k=1}^{3}\frac{\{\lambda(\lambda+s\beta+s\rho^k)\}^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\rho^k)(\lambda+s+s\rho^k)}\right.$$

$$+\ \beta(1-\beta)^2\left[\frac{\lambda+s\beta+s\rho}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho)}\times\sum_{k=1}^{2}\frac{\{\lambda(\lambda+s\beta+s\rho^k+s\rho^{k+1})\}^2}{(\lambda+s\rho^k+s\rho^{k+1})(\lambda+s+s\rho^k+s\rho^{k+1})}\right.$$

$$+\ \frac{(\lambda+s\beta+s\rho^2)}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho^2)}\ \frac{\{\lambda(\lambda+s\beta+s\rho+s\rho^3)\}^2}{(\lambda+s\rho+s\rho^3)(\lambda+s+s\rho+s\rho^3)}$$

$$+\ (1-\beta)^3\ \frac{(\lambda+s\beta+s\rho)(\lambda+s\beta+s\rho+s\rho^2)\{\lambda(\lambda+s\beta+s\rho+s\rho^2+s\rho^3)\}^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho)(\lambda+s\beta+s+s\rho+s\rho^2)}$$

$$\left.\times\ \frac{1}{(\lambda+s\rho+s\rho^2+s\rho^3)(\lambda+s+s\rho+s\rho^2+s\rho^3)}\right]\ ;\ (4.8)$$

For the sum T_5 we get the transform

$$\psi_5(s,0)=\left(\frac{\lambda}{\lambda+s\beta}\right)^4\left\{\beta^4\ \frac{\lambda}{\lambda+s}+\beta^3(1-\beta)\sum_{k=1}^{4}\frac{[\lambda(\lambda+s\beta+s\rho^k)]^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\rho^k)(\lambda+s+s\rho^k)}\right.$$

$$+\ \beta^2(1-\beta)^2\left[\sum_{k=2}^{4}\left\{\frac{(\lambda+s\beta+s\rho^{k-1})[\lambda(\lambda+s\beta+s\rho+s\rho^k)]^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho^{k-1})}\right.\right.$$

$$\left.\times\ \frac{1}{(\lambda+s\rho+s\rho^k)(\lambda+s+s\rho+s\rho^k)}\right\}$$

$$+\ \sum_{k=3}^{4}\left\{\frac{(\lambda+s\beta+s\rho^{k-2})[\lambda(\lambda+s\beta+s\rho^2+s\rho^k)]^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho^{k-2})(\lambda+s\rho^2+s\rho^k)}\right.$$

$$\left.\times\ \frac{1}{\lambda+s+s\rho^2+s\rho^k}\right\}$$

$$+\ \frac{(\lambda+s\beta+s\rho)[\lambda(\lambda+s\beta+s\rho^3+s\rho^4)]^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho)(\lambda+s\rho^3+s\rho^4)}$$

$$\left.\times\ \frac{1}{\lambda+s+s\rho^3+s\rho^4}\right]$$

$$+\ \beta(1-\beta)^3\left[\frac{\lambda+s\beta+s\rho}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho)(\lambda+s\beta+s+s\rho+s\rho^2)}\times\right.$$

$$
\left\{\frac{(\lambda+s\beta+s\rho+s\rho^2)[\lambda(\lambda+s\beta+s\rho+s\rho^2+s\rho^3)]^2}{(\lambda+s\rho+s\rho^2+s\rho^3)(\lambda+s+s\rho+s\rho^2+s\rho^3)}\right.
$$

$$
+\frac{(\lambda+s\beta+s\rho+s\rho^2)[\lambda(\lambda+s\beta+s\rho^2+s\rho^3+s\rho^4)]^2}{(\lambda+s\rho^2+s\rho^3+s\rho^4)(\lambda+s+s\rho^2+s\rho^3+s\rho^4)}
$$

$$
\left.+\frac{(\lambda+s\beta+s\rho^2+s\rho^3)[\lambda(\lambda+s\beta+s\rho+s\rho^3+s\rho^4)]^2}{(\lambda+s\rho+s\rho^3+s\rho^4)(\lambda+s+s\rho+s\rho^3+s\rho^4)}\right\}
$$

$$
+\frac{(\lambda+s\beta+s\rho^2)(\lambda+s\beta+s\rho+s\rho^3)[\lambda(\lambda+s\beta+s\rho+s\rho^2+s\rho^4)]^2}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho^2)(\lambda+s\beta+s+s\rho+s\rho^3)}
$$

$$
\left.\times\frac{1}{(\lambda+s\rho+s\rho^2+s\rho^4)(\lambda+s+s\rho+s\rho^2+s\rho^4)}\right]
$$

$$
+(1-\beta)^4\left\{\frac{(\lambda+s\beta+s\rho)(\lambda+s\beta+s\rho+s\rho^2)(\lambda+s\beta+s\rho+s\rho^2+s\rho^3)}{(\lambda+s\beta)(\lambda+s\beta+s)(\lambda+s\beta+s+s\rho)(\lambda+s\beta+s+s\rho+s\rho^2)}\right.
$$

$$
\times\frac{[\lambda(\lambda+s\beta+s\rho+s\rho^2+s\rho^3+s\rho^4)]^2}{(\lambda+s\beta+s+s\rho+s\rho^2+s\rho^3)(\lambda+s\rho+s\rho^2+s\rho^3+s\rho^4)}
$$

$$
\left.\times\frac{1}{\lambda+s+s\rho+s\rho^2+s\rho^3+s\rho^4}\right\}. \qquad (4.9)
$$

The pattern in these results is fairly evident, but it is clear that the Laplace transform of the intensity function (Cox and Lewis, 1966, Ch. 5),

$$
m_f^*(s) = \sum_{r=1}^{\infty} \psi_r(s;0)
$$

is not obtainable. This is disappointing in view of the simplicity of the result for the EMA1 process (Lawrance and Lewis, 1975). It is probably true that as much information can be obtained from the higher order joint moments given in Section 3 as can be obtained from the intensity function. As a particular case,

for the EMA1 process, the intensity function differentiates the cases β and $1-\beta$, where the serial correlations do not. However, a direct estimate of β is obtainable from $C_{1,2}(1)$ and $C_{2,1}(1)$ which are given at (3.1) and (3.2).

Some idea of the behaviour of the intensity function can, however, be obtained. The limiting behaviour is discussed in Section 6. Consider now the value of the intensity function at 0.

Note that

$$\mathrm{pr}(X_1 + \ldots + X_n \le t) \le \mathrm{pr}(\epsilon_1 + \ldots + \epsilon_n \le \tfrac{t}{\beta})$$

$$= \sum_{k=n}^{\infty} e^{-\lambda(\frac{t}{\beta})} \frac{\{\lambda(\frac{t}{\beta})\}^k}{k!} . \qquad (4.10)$$

Since X_1 is exponentially distributed, we have

$$\mathrm{pr}(N_t \ge 1) = \mathrm{pr}(X_1 \le t) = (1-e^{-\lambda t})$$

and, from (4.10)

$$\frac{\sum_{k=2}^{\infty} \mathrm{pr}(N_t \ge k)}{t} \le \frac{\frac{\lambda}{\beta} t - \left(1-e^{-\frac{\lambda}{\beta} t}\right)}{t}$$

$$= \frac{\lambda}{\beta} + \frac{(e^{-\frac{\lambda}{\beta} t} - 1)}{t} ,$$

which tends to 0 as $t \to 0$.

Now the intensity function is the derivative of the function $E(N_t)$.

Hence, we have

$$\lim_{t \downarrow 0} \frac{E(N_t)}{t} = \lim_{t \downarrow 0} \frac{\sum_{k=1}^{\infty} pr(N_t \geq k)}{t} = \lim_{t \downarrow 0} \frac{1 - e^{-\lambda t}}{t} = \lambda ;$$

i.e., that the derivative of the mean value function (the intensity function) at $t = 0$ is λ.

5. THE VARIANCE-TIME CURVE AND SEQUENCE

We can easily obtain the variance function for T_r and the index of dispersion for intervals (Cox and Lewis, 1966, p. 71), from the results of the previous sections.

For the variance function we have

$$\mathrm{Var}(T_r) = r\,\mathrm{Var}(X_1) + 2\sum_{i=1}^{r-1}(r-i)\{E(X_1X_{i+1}) - E(X_1)E(X_{i+1})\}$$

$$= \frac{1}{\lambda^2}\{r + 2\sum_{i=1}^{r-1}(r-i)[(1-\beta)\rho^{i-1}\{\beta(1-\rho) + (1-\beta)\rho\}]\}$$

$$= \frac{1}{\lambda^2}\{r + 2r(1-\beta)[\beta(1-\rho) + (1-\beta)\rho]\sum_{k=1}^{r-1}\rho^{i-1}$$

$$- 2(1-\beta)[\beta(1-\rho) + (1-\beta)\rho]\sum_{i=1}^{r-1} i\rho^{i-1}\}$$

$$= \frac{1}{\lambda^2}[r + 2(1-\beta)\{\beta(1-\rho) + (1-\beta)\rho\}\{r\sum_{j=0}^{r-2}\rho^j - \sum_{i=1}^{r-1} i\rho^{i-1}\}].$$

If $\rho < 1$

$$\mathrm{Var}(T_r) = \frac{1}{\lambda^2}[r + 2(1-\beta)\{\beta(1-\rho) + (1-\beta)\rho\}\{r\,\frac{1-\rho^{r-1}}{1-\rho} - \frac{d}{d\rho}\frac{1-\rho^r}{1-\rho}\}]$$

$$= \frac{1}{\lambda^2}[r + 2(1-\beta)\{\beta(1-\rho) + (1-\beta)\rho\}\{\frac{r}{1-\rho} - \frac{(1-\rho^r)}{(1-\rho)^2}\}]. \tag{5.1}$$

Hence, for $\rho < 1$ the normalized variance sequence is

$$J_r = \frac{\mathrm{Var}\{T_r\}}{rE[T_1]^2} = 1 + 2(1-\beta)\{\beta(1-\rho) + (1-\beta)\rho\}\{\frac{1}{1-\rho} - \frac{1-\rho^r}{r(1-\rho)^2}\}\ .$$

Therefore, for $\rho < 1$ the index of dispersion is

$$J = \lim_{r\to\infty} J_r = 1 + 2(1-\beta)\{\beta(1-\rho) + (1-\beta)\rho\}\,\frac{1}{(1-\rho)}\ . \tag{5.2}$$

For $\rho = 1$

$$\mathrm{Var}\{T_r\} = \frac{1}{\lambda^2}[r + 2(1-\beta)^2 \sum_{i=1}^{r-1} (r-i)].$$

Hence

$$\lim_{n\to\infty} \frac{\mathrm{Var}\{T_r\}}{rE[T_1]^2} = \lim_{r\to\infty} 1 + 2(1-\beta)^2 \frac{\sum_{k=1}^{r-1} k}{r} = \infty.$$

Thus the process is overdispersed relative to the Poisson process for which $\lim_{r\to\infty} J_r = 1$. In particular, if an observed process had exponentially distributed marginal distributions for the X_i but $\lim_{r\to\infty} J_r$ was much greater than one, the EARMA (1,1) process could be a candidate as a model.

As a byproduct of (5.2), for $\rho < 1$ we get values for the slope of the variance time curve, $\lim_{t\to\infty} V'(t) = V'(\infty)$, the initial points of the spectrum of counts $g_+(\omega)$ and the spectrum of intervals $f_+(\omega)$ as

$$f_+(0+) = \frac{1}{\pi} \lim_{r\to\infty} J_r, \tag{5.3}$$

$$V'(\infty) = \lambda \lim_{r\to\infty} J_r, \tag{5.4}$$

$$g_+(0+) = \frac{\lambda}{\pi} \lim_{r\to\infty} J_r, \tag{5.5}$$

for $\rho < 1$, (Cox and Lewis, 1966, 4.6.12).

The relationship $f_+(0+)$ can be verified directly from (2.5); (5.5) says, using (5.2), that the initial point of the spectrum $g_+(\omega)$ is greater than its value $\frac{\lambda}{\pi}$ for a Poisson process.

6. LIMIT THEOREMS

In this section we will study the mixing properties of $\{X_n\}$, obtain strong laws and central limit theorems for $\{T_n\}$ and $\{N_t\}$, and obtain renewal type results for $E[N_t]$. We will assume throughout this section that $\rho < 1$. Our technique usually will be to use the limiting behavior of related Markov chains and Markov renewal processes to obtain limiting results for the EARMA (1,1) process. The Markov chain and Markov renewal process will have the real line as state space. Although it is usually hard to obtain specific results concerning the limiting behavior of these processes if they have a nondiscrete state space, the linear forms of (1.1) and (1.2) allow us to obtain the limiting behavior quite easily in this instance.

6.1 Mixing properties and asymptotic normality

We first consider the limiting distribution of the sequence $\{X_n\}$ in the case when A_0 is not exponential (λ). From (1.1) it follows that the dependence of X_n on X_{n-1} is through the auto-regressive sequence $\{A_n\}$. Since $A_n = \rho A_{n-1} + V_n \epsilon_n$ $(n=1,2,\ldots)$, $\{A_n;\ n=0,1,\ldots\}$ is a Markov chain (a discrete time parameter process which satisfies the first-order Markov property), with state space $(\mathbb{R}_+, \underline{\mathcal{B}}_+)$, where $\mathbb{R}_+ = [0,\infty)$ and $\underline{\mathcal{B}}_+$ denotes the collection of Borel subsets of $\mathbb{R}_+$. The transition function $P(x,B) = \mathrm{pr}\{A_n \in B \mid A_{n-1} = x\}$ for the Markov chain is given by

$$P(x,B) = \rho\varepsilon_{\rho x}(B) + (1-\rho)\int_{B-\rho x} \lambda e^{-\lambda y}\, dy \tag{6.1}$$

for any interval B of $\mathbb{R}_+$ and $x \geq 0$, where

$$\varepsilon_{\rho x}(B) = \begin{cases} 1 & \text{if} \quad \rho x \epsilon B, \\ 0 & \text{otherwise} \end{cases}$$

and for $z \geq 0$, $B - z = \{y-z: y \epsilon B\}$.

Thus, A_n has, conditionally, a shifted exponential distribution with an atom at the point ρx. By (1.2) we have the expansion

$$A_k = \rho^k A_0 + \rho^{k-1} V_1 \epsilon_1 + \ldots + \rho V_{k-1} \epsilon_{k-1} + V_k \epsilon_k,$$

and it is not hard to show that

$$E[\exp\{-s(V_k \epsilon_k + \rho V_{k-1} \epsilon_{k-1} + \ldots + \rho^{k-1} V_1 \epsilon_1)\}] = \frac{\lambda + s\rho^k}{\lambda + s},$$

$$= \rho^k + (1-\rho^k)\frac{\lambda}{\lambda+s}.$$

Hence, for the $k^{\underline{th}}$ order transition function we have

$$P^k(x,B) = \operatorname{pr}\{A_k \epsilon B \mid A_0 = x\}$$

$$= \rho^k \varepsilon_{\rho^k x}(B) + (1-\rho^k) e^{\lambda \rho^k x} \int_B \lambda e^{-\lambda y} dy\,, \tag{6.2}$$

for $k \geq 1$, $x \geq 0$, and any Borel set B in $\mathbb{R}_+$. Taking limits as $k \to \infty$ we have

$$\lim_{k \to \infty} P^k(x,B) = \pi(B) \equiv \int_B \lambda e^{-\lambda y} dy\,,$$

which is simply an exponential distribution.

From (1.1), adding on the moving average operation to get the complete $\{X_n\}$ process

$$P\{X_n \epsilon B \mid A_0 = x\} = \beta \operatorname{pr}\{\beta \epsilon_n \epsilon B\} + (1-\beta)\operatorname{pr}\{\beta \epsilon_n + A_{n-1} \epsilon B \mid A_0 = x\}$$

$$= \beta \pi(B\beta^{-1}) + (1-\beta) \int \pi(dy) P^{n-1}(x, B-\beta y).$$

where $B\beta^{-1} = \{y\beta^{-1} : y \in B\}$. Hence, taking limits

$$\lim_{n\to\infty} P\{X_n \in B \mid A_0 = x\} = \beta\pi(B\beta^{-1}) + (1-\beta)\int \pi(dy)\pi(B-\beta y)$$
$$= \pi(B).$$

Thus we have shown that if we start generating the sequence $\{X_n\}$ with a random variable A_0 that has an arbitrary (and possibly degenerate) distribution, the distribution function of X_n will converge to the exponential distribution as $n \to \infty$.

We will now study the mixing properties of $\{X_n\}$ when A_0 is exponentially distributed. Let $\underline{\underline{F}}_m$ be the σ-algebra generated by the random variables $A_0,\ldots,A_m$, $\epsilon_1,\ldots,\epsilon_m$, and $U_1,\ldots,U_m$. Let $\underline{\underline{G}}_m$ be the σ-algebra generated by $A_m, A_{m+1},\ldots;$ $\epsilon_m,\epsilon_{m+1},\ldots;$ and $U_m, U_{m+1},\ldots$. Let $L^2(\underline{\underline{F}}_m)$, (respectively $L^2(\underline{\underline{G}}_m)$), be the collection of real-valued functions that are measurable with respect to $\underline{\underline{F}}_m$, (respectively $\underline{\underline{G}}_m$), and have finite second moments. Since $\{\epsilon_n\}$ and $\{U_n\}$ are independent sequences of random variables, by the Markov property of $\{A_n\}$ we have for $f \in L^2(\underline{\underline{F}}_m)$ and $g \in L^2(\underline{\underline{G}}_{m+k})$, $k \geq 1$, that

$$\begin{aligned} E(fg) - E(f)E(g) &= E[f\{E(g \mid A_m) - E(g)\}] \\ &= E[f\int E(g \mid A_{m+k} = y)\{P^k(A_m,dy) - \pi(dy)\}] \\ &= \rho^k E[f\int E(g \mid A_{m+k} = y)\{\varepsilon_{\rho^k A_m}(dy) - \pi(dy)\}] \\ &\quad + (1-\rho^k)E\{f\int E(g \mid A_{m+k} = y)(e^{\lambda\rho^k A_m} - 1)\pi(dy)\} \end{aligned} \tag{6.3}$$

by (6.2).

By Hölder's inequality

$$E\{f\int E(g \mid A_{m+k} = y)\varepsilon_{\rho^k A_m}(dy)\}$$

$$= E\{fE(g \mid A_{m+k} = \rho^k A_m)\}$$

$$\le E(f^2)^{1/2} E[\{E(|g| \mid A_{m+k} = \rho^k A_m)\}^2]^{1/2}.$$

$$\le E(f^2)^{1/2} E\{E(g^2 \mid A_{m+k} = \rho^k A_m)\}^{1/2}.$$

But, letting $z = \rho^k y$

$$E\{E(g^2 \mid A_{m+k} = \rho^k A_m)\}$$

$$= \int_0^\infty \lambda e^{-\lambda y} E(g^2 \mid A_{m+k} = \rho^k y)dy$$

$$= \frac{1}{\rho^k}\int_0^\infty \lambda e^{-z\lambda\rho^{-k}} E(g^2 \mid A_{m+k} = z)dz$$

$$\le \frac{1}{\rho^k} E(g^2),$$

if $0 < \rho < 1$.

Thus, for $0 \le \rho < 1$

$$\rho^k E[f\int E(g \mid A_{m+k} = y)\{\varepsilon_{\rho^k A_m}(dy) - \pi(dy)\}]$$

$$\le \rho^{\frac{k}{2}} E(f^2)^{1/2} E(g^2)^{1/2} - \rho^k E(f)E(g). \qquad (6.4)$$

Using the Hölder inequality again we have

$$E\{f\int E(g \mid A_{m+k} = y)(e^{\lambda\rho^k A_m} - 1)\pi(dy)\}$$

$$= E(g)E\{f(e^{\lambda\rho^k A_m} - 1)\}$$

$$\le E(g)E(f^2)^{1/2} E\{(e^{\lambda\rho^k A_m} - 1)^2\}^{1/2}.$$

For k sufficiently large, and $0 < \rho < 1$

$$E\{(e^{\lambda\rho^k A_m} - 1)^2\} = E(e^{2\lambda\rho^k A_m} - 2e^{\lambda\rho^k A_m} + 1)$$

$$= \frac{\lambda}{\lambda(1+2\rho^k)} - 2\frac{\lambda}{\lambda(1+\rho^k)} + 1$$

$$= \frac{1+\rho^k - 2(1+2\rho^k) + (1+2\rho^k)(1+\rho^k)}{(1+2\rho^k)(1+\rho^k)}$$

$$= \frac{2\rho^{2k}}{1+3\rho^k+2\rho^{2k}}$$

$$\le 2\rho^{2k}.$$

Hence from (6.3) and (6.4)

$$E(fg) - E(f)E(g) \le E(f^2)^{1/2}E(g^2)^{1/2}(\rho^{\frac{k}{2}}+\rho^k+\sqrt{2}\,\rho^k)$$

$$\le E(f^2)^{1/2}E(g^2)^{1/2}\rho^{\frac{k}{2}}(2+\sqrt{2}). \tag{6.5}$$

Therefore, when A_0 is exponentially distributed the sequence $\{(A_n,\epsilon_n,U_n)\}$ is strong mixing in the sense of Rosenblatt (1971) and is in fact asymptotically uncorrelated in the sense of Rosenblatt (1971). By (1.1) $\{X_n\}$ is also strong mixing and asymptotically uncorrelated. Further, for any measurable sets $B = f(X_1,\ldots,X_m)$ and $C = g(X_{m+k},X_{m+k+1},\ldots)$ for suitable functions f and g, by (6.5)

$$|pr(B \cap C) - pr(B)pr(C)| \le (2+\sqrt{2})\rho^{\frac{k}{2}}. \tag{6.6}$$

In addition it follows that

$$\lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{\lambda} \quad \text{a.s.} \tag{6.7}$$

by the strong law for a stationary sequence of random variables; cf. Doob (1953, Chapter 10). Also since

$$\frac{T_{N_t}}{N_t} \le \frac{t}{N_t} \le \frac{T_{N_t+1}}{N_t} \tag{6.8}$$

we have

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \qquad \text{a.s..} \tag{6.9}$$

To obtain a central limit theorem for $\{T_n\}$ we note that, using (5.2), the hypothesis of Theorem (20.1) in Billingsley (1968) is satisfied by (6.6) with

$$\begin{aligned}\sigma^2 &= \text{Var } X_1 + 2\sum_{n=1}^{\infty} \text{cov}(X_1, X_{1+n}) \\ &= \frac{1}{\lambda}\,[1 + 2(1-\beta)\{\beta(1-\rho) + (1-\beta)\rho\}\,\frac{1}{1-\rho}],\end{aligned} \tag{6.10}$$

which is positive. Hence,

$$\lim_{n\to\infty} \text{pr}\left\{\frac{T_n - n\frac{1}{\lambda}}{\sqrt{n}\,\sigma} \le x\right\} = \Phi(x), \tag{6.11}$$

where

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy.$$

By the remark after the proof of Theorem (17.3) in Billingsley (1968) we also have

$$\lim_{t\to\infty} \text{pr}\left\{\frac{N_t - \lambda t}{\sigma\lambda^{3/2} t^{1/2}} \le x\right\} = \Phi(x), \tag{6.12}$$

i.e., the counting process, suitably normalized is, in the limit, asymptotically normally distributed.

Equations (6.7) and (6.11) show that the estimate $\bar{X} = \frac{T_n}{n}$ of $E[X_n]$ is strongly consistent and asymptotically normally distributed with variance $\frac{1}{n}\sigma^2$, with σ^2 defined by (6.10). Also

(6.9) and (6.12) show that the estimate $\hat{\lambda} = \frac{N_t}{t}$ of λ is also strongly consistent and asymptotically normally distributed with variance $\frac{\sigma^2\lambda^3}{t}$.

6.2 Renewal type theorems

We will now turn our attention to renewal type limit theorems for $E(N_t)$ as $t \to \infty$.

For any Borel set B in $\mathbb{R}_+$, $t \geq 0$, and $x \geq 0$, let

$$Q(x,B,t) = \mathrm{pr}(A_1 \in B, T_1 \leq t \mid A_0 = x) . \tag{6.13}$$

From the definition of the process (1.1) and (1.2) we have

$$\begin{aligned} Q(x,B,t) = \rho\, \varepsilon_x(B)\,[\beta\{1-\exp(-\lambda\beta^{-1}t)\} \\ + (1-\beta)[1-\exp\{-\lambda\beta^{-1}\max(t-x,0)\}]] \\ + (1-\rho)\int_0^\infty \lambda e^{-\lambda y}\, dy\; I_B(\rho x+y)\{\beta I_{[0,t]}(\beta y) \\ + (1-\beta)I_{[0,t]}(\beta y+x)\}, \end{aligned}$$

where

$$I_B(z) = \begin{cases} 1 & \text{if} \quad z \in B, \\ 0 & \text{otherwise.} \end{cases}$$

By (1.1) and (1.2), $\{(A_n,T_n); n = 0,1,\ldots\}$ is a Markov renewal process in the sense of Çinlar (1975, Ch. 10), with state space $(R_+,\underset{\sim}{R}_+)$ and semi-Markov kernel Q defined in (6.13); that is, letting $T_0 = 0$,

$$\mathrm{pr}(A_{n+1} \in B, T_{n+1} - T_n \leq t \mid A_0,\ldots,A_n,T_0,\ldots,T_n) \tag{6.14}$$

$$= Q(A_n,B,t).$$

Thus the T_n process of the EARMA1 process is a marginal process in a Markov renewal process with a non-discrete state space. Let

$$R(x,B,t) = \sum_{n=0}^{\infty} Q^n(x,B,t), \tag{6.15}$$

where

$$Q^0(x,B,t) = \varepsilon_x(B)\varepsilon_0([0,t]) \tag{6.16}$$

and

$$Q^n(x,B,t) = \int_0^t \int_0^\infty Q(x,dy,ds)Q^{n-1}(y,B,t-s)$$

for $n \geq 1$, $x \geq 0$, $t \geq 0$, and Borel set B. Note that

$$E(N_t \mid A_0 = x) = R(x,E,t). \tag{6.17}$$

Since $\{\epsilon_n\}$ is a sequence of independent exponentially distributed random variables, for any Borel set B with positive Lebesque measure

$$\mathrm{pr}(A_n \in B \text{ infinitely often} \mid A_0 = x) = 1$$

for all $x \geq 0$. Therefore, condition A of Jacod (1971, p. 86) is satisfied. Further, since

$$\mathrm{pr}(A_n \in B,\ T_n \leq t \mid A_0 = x)$$
$$\geq \beta^n \rho^{n-1}(1-\rho)\mathrm{pr}(\epsilon_n \in B - \rho^n x, \epsilon_1 + \ldots + \epsilon_n \leq t\beta^{-1}),$$

for $n \geq 2$, $a > 0$, and each $\delta > 0$, there is a Borel set C of R_+ of positive Lebesgue measure so that the density of $Q^n(x,\cdot,(a-\delta,a+\delta))$ with respect to Lebesgue measure has a positive lower bound on the product set $C \times C$. Hence, the hypotheses of

Jacod (1971, Th. 3, p. 107) are satisfied and

$$\lim_{t\to\infty} R(x,B,t+C) = \lim_{t\to\infty} \int_C R(x,B,t+du) = \frac{1}{m}\pi(B)|C| \tag{6.18}$$

for any Borel sets B and C of $\mathbb{R}_+$ and $x \ge 0$, where $|C|$ denotes the Lebesgue measure of C, and

$$m = \int_0^\infty \pi(dy) \int_0^\infty \{1 - Q(y,\mathbb{R}_+,s)\}ds$$

$$= \int_0^\infty pr(T_1>s)\,ds = \frac{1}{\lambda} \,. \tag{6.19}$$

Finally taking $C = [0,h]$ and $B = \mathbb{R}_+$ in (6.18) we have

$$\lim_{t\to\infty}[E(N_{t+h} \mid A_0=x) - E(N_t \mid A_0=x)] = \lambda h, \tag{6.20}$$

for all $x \ge 0$. This is a Blackwell-type renewal theorem for the EARMA(1,1) process.

6.3 The intensity function

We will now show that the intensity function for the counting process exists and its limit as $t \to \infty$ is λ. Let

$$q(x,t) = \lim_{h\downarrow 0} \frac{1}{h}\{Q(x,\mathbb{R}_+,t+h) - Q(x,\mathbb{R}_+,t)\} \tag{6.21}$$

$$= \begin{cases} \lambda\exp\{-\lambda\beta^{-1}t\} & \text{for} \quad t < x, \\ \lambda\exp\{-\lambda\beta^{-1}t\} + (1-\beta)\lambda\beta^{-1}\exp\{-\lambda\beta^{-1}(t-x)\} & \text{for} \quad t \ge x. \end{cases}$$

Consider the Markov renewal equation, (MRE),

$$f(x,t) = q(x,t) + \int_0^t \int_0^\infty Q(x,dy,ds)f(y,t-s). \tag{6.22}$$

Since $q(x,t) \leq 1$ and $\sup_x Q(x,R_+,t) < 1$ for $t > 0$, by the results of Çinlar (1975, Ch. 10, Sec. 3), (6.22) has the unique solution

$$m(x,t) = \int_0^t \int_0^\infty R(x,dy,ds)q(y,t-s).$$

The function $U(t) = \int_0^t m(u)du$ satisfies the MRE,

$$f(x,t) = Q(x,R_+,t) + \int_0^t \int_0^\infty Q(x,dy,ds)f(y,t-s)$$

which has as its unique solution $R(x,R_+,t) - 1$, where $R(x,R_+,t)$ is defined by (6.15). Therefore, $m(x,t)$ is the intensity of the counting process evaluated at t given that $A_0 = x$.

Note that $\int_0^\infty \pi(dx)q(x,t) = \lambda e^{-\lambda t}$, and for each $x \in R_+$ $t \to q(x,t)$ is monotone decreasing. Hence, for $b > 0$.

$$\sigma_1' \equiv b \sum_n^\infty \int_0^\infty \pi(dx)\sup\{q(x,t): \ nb \leq t < (n+1)b\}$$
$$\leq b\int_0^\infty \pi(dx)q(x,0) + b\int_0^\infty \pi(dx)\int_0^\infty q(x,t)dt < \infty$$

Hence, q is directly Riemann integrable with respect to π in the sense of Çinlar (1969, p. 388). By (6.18), the hypotheses of the Theorem on page 390 of Çinlar (1969) are satisfied, and we have

$$\lim_{t\to\infty} m(x,t) = \lambda$$

for all $x \geq 0$. <u>Thus the intensity function goes to</u> λ <u>as</u> $t \to \infty$.

The same types of techniques, i.e. writing a Markov renewal equation, writing down its solution, and computing its limit as $t \to \infty$, can be used to show that the distribution of the forward recurrence time is in the limit as $t \to \infty$ exponential with parameter λ.

Note that subsequent intervals are, in the limit, not exponentially distributed. Following Cox and Lewis (1966, Ch. 4) call these intervals $L_1, L_2, \ldots$. Their marginal and joint distributions can, in theory, be obtained from results of Section 3 using Palm-Khinchine relationships. We note only that their means are, (Cox and Lewis, 1966, Ch. 4),

$$E[L_i] = E[X_n]\{1 + C^2(X_n)\rho(i)\}, \qquad (i = 1,2,\ldots).$$

Since $E[X_n] = \frac{1}{\lambda}$ and $C^2(X_n) = 1$, using the expressions (2.3) and (2.4) for $\rho(i)$ we get

$$E[L_i] = \frac{1}{\lambda}\{1 + c(\beta,\rho)\rho^{i-1}\}.$$

This decays geometrically to $\frac{1}{\lambda}$. When $\rho = 0$ and we have only the two-dependent EMA1 process, the bias does not extend beyond $E[L_1]$, as would be expected from the construction of the process.

7. EXTENSIONS AND CONCLUSIONS

The EARMA (1,1) process described here is a generalization of the EAR1 and EMA1 processes and has the correlation structure of a mixed-moving average-autoregressive process of order (1,1) and an exponential marginal distribution. It thus provides an alternative to an i.i.d. exponential sequence or a Poisson process which is relatively simple probabilistically and has an easily visualized dependency structure. Moreover, it is easy to generate on a computer in simulations, and asymptotic properties follow easily from the observation that it is the marginal process in a Markov renewal process with a nondiscrete state space.

Lewis (1975) discusses methods of generating similar processes in which either (i) the marginal distribution of the X_i's is Gamma, mixed exponential or Weibull; and/or (ii) the dependency structure is that of higher order ARMA processes. We do not give details here; probabilistic properties of these processes, as well as multivariate processes, will be discussed in later papers.

Similarly estimation properties are discussed elsewhere; we note only that results such as those in Section 3 allow one to compute asymptotic variances of serial correlations and other statistics based on the X_i's. Thus, we have a tractable process with which to examine the efficacy of the many tests for renewal and Poisson processes given in Cox and Lewis (1966, Ch. 6).

A limitation of the process is that the serial correlations are all positive and, for $\rho = 0$, i.e. the EMA1 process, are bounded above by 1/4.

Thus, the question arises as to whether there are processes with exponential marginal distributions and ARMA (1,1) second order correlation structure and which cover a broader range than the EARMA (1,1) process, though perhaps at a cost of more complicated structure.

One such process is now discussed briefly (Cox and Lewis, 1966, pp. 7, 194-204). In this two state semi-Markov model there are two types of intervals with p.d.f.s. $f_1(x)$ and $f_2(x)$ sampled in accordance with a two-state Markov chain for which

$$P = \begin{pmatrix} \alpha_1 & 1-\alpha_1 \\ 1-\alpha_2 & \alpha_2 \end{pmatrix} \quad (7.1) \qquad \text{and} \qquad \underline{\Pi} = \underline{\Pi}\ \underline{P} = \left(\frac{1-\alpha_2}{2-\alpha_1-\alpha_2}\ ,\ \frac{1-\alpha_1}{2-\alpha_1-\alpha_2}\right). \quad (7.2)$$

It is assumed that no knowledge is available about the type of interval. Then, the distribution of an interval X_k is

$$f_X(x) = \pi_1 f_1(x) + \pi_2 f_2(x) \tag{7.3}$$

and the correlation between X_1 and X_{k+1} is

$$\rho(k) = A\ \beta^k \qquad (k = 1,2,\ldots), \tag{7.4}$$

where A is a positive constant and $\beta = \alpha_1 + \alpha_2 - 1 = \alpha_1 - (1-\alpha_2)$. Thus the correlation structure is that of an ARMA (1,1) process.

If we let

$$f_1(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}}{1-e^{-\lambda x_0}} & 0 \le x \le x_0, \\ 0 & x > x_0; \end{cases}$$

$$f_2(x) = \begin{cases} 0 & x \le x_0, \\ \dfrac{\lambda e^{-\lambda x}}{e^{-\lambda x_0}} & x > x_0; \end{cases}$$

the marginal distribution of an interval X, $f_X(x)$, is exponential(λ) if we set $\pi_1 = 1 - \exp(-x_0)$. There is then only one degree of freedom left in the matrix $\underline{P}$, and in addition to λ, we have free parameters π_1 (or x_0) and α_1. What then is the range of β, and can it be negative?

Straightforward manipulation shows that

$$\beta = \frac{\pi_1 - \alpha_1}{\pi_1 - 1},$$

which lies between zero and one. Thus the model is no broader than the EARMA (1,1) model and the question of obtaining negative correlation is still open. More complicated non-linear schemes for doing this are discussed by Lewis (1975).

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